



# UK Maths Trust

## British Mathematical Olympiad

### Round 2

Wednesday 24 January 2024

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### Instructions

1. Do not turn over until the invigilator tells you to do so.
2. Time allowed:  $3\frac{1}{2}$  hours.
3. **Full written solutions – not just answers – are required**, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then write up your best attempt.
4. **You may hand in rough work** where it contains calculations, examples or ideas not present in your final attempt; write 'ROUGH' at the top of each page of rough work.
5. **Each question carries 10 marks**. One or two complete solutions will gain far more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all the problems.
6. The use of rulers and compasses is allowed, but **calculators and protractors are forbidden**. You are strongly encouraged to use geometrical instruments to construct large, accurate diagrams for geometry problems.
7. Start each question on an official answer sheet on which there is a **QR code**.
8. If you use additional sheets of (plain or lined) paper for a question, please write the following in the top left-hand corner of each sheet. (i) The question number. (ii) The page number for that question. (iii) The digits following the ':' from the question's answer sheet QR code. **Please do not write your name or initials on any additional sheets.**
9. **Write on one side of the paper only**. Make sure your writing and diagrams are not too faint. (Your work will be scanned for marking.)
10. **Arrange your answer sheets, including rough work, in question order before they are collected**. If you are not submitting work for a particular problem, remove the associated answer sheet.
11. To accommodate candidates in other time zones, please do not discuss any aspect of the paper on the internet until **8am GMT on Friday 26 January** when the solutions video will be released as <https://bmos.ukmt.org.uk>. Candidates in time zones more than 3 hours ahead of GMT must sit the paper on Thursday 25 January (as defined locally).
12. Around 24 high-scoring students eligible to represent the UK at the International Mathematical Olympiad will be invited to a training session held in Cambridge around the Easter holidays.

Enquiries about the British Mathematical Olympiad should be sent to: [challenges@ukmt.org.uk](mailto:challenges@ukmt.org.uk)

1. In the sequence 7, 76, 769, 7692, 76923, 769230, ..., the  $n$ th term is given by the first  $n$  digits after the decimal point in the expansion of  $10/13 = 0.7692307692\dots$

Prove that of the first 60 terms of the sequence, at least 49 have three or more prime factors (repeated prime factors are allowed; for example,  $76 = 2 \times 2 \times 19$  has three prime factors).

2. Find all functions  $f$  from the integers to the integers such that for all integers  $n$ :

$$2f(f(n)) = 5f(n) - 2n.$$

3. Let  $ABC$  be an acute-angled triangle with  $AB > AC$ . Let  $P$  be the intersection of the tangents to the circumcircle of  $ABC$  at  $B$  and  $C$ . The line through the midpoints of line segments  $PB$  and  $PC$  meets lines  $AB$  and  $AC$  at  $X$  and  $Y$  respectively.

Prove that the quadrilateral  $AXPY$  is cyclic.

4. Let  $m < n$  be positive integers. Start with  $n$  piles, each of  $m$  objects. Repeatedly carry out the following operation: choose two piles and remove  $n$  objects in total from the two piles. For which  $(m, n)$  is it possible to empty all the piles?